A NEW PROOF OF A. F. TIMAN'S APPROXIMATION THEOREM

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ABSTRACT

This paper gives a new proof of A. F. Timan's approximation theorem. It seems to be of considerable advantage that for a fixed n our polynomial $G_n(f)$ is of degree $\leqq n$-1 and depends on n values of $f(x)$ only.

1. Introduction

Let f be a continuous function on $[-1, +1]$ with the modulus of continuity $\omega(h)$. It was discovered by Nikol'skii [4] that the quality of approximation by polynomials increases toward the end points of the interval. Later, Timan [5] obtained the following result.

THEOREM 1. *There exists a constant M such that for each function $f \in C[-1, +1]$ there is a sequence of polynomials $P_n(x)$ for which*

$$(1.1) \qquad |f(x) - P_n(x)| \leqq M\left[\omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right].$$

By showing (1.1), we give a new proof of A. F. Timan's theorem. It seems to be of considerable advantage that for a fixed n our polynomial $G_n(f)$ (see below) is of degree $\leqq n-1$ and depends on n values of $f(x)$ only. Compare also with known results [1] and [5].

Let

$$(1.2) \qquad x_{kn} = \frac{\cos(2k-1)\pi}{2n} = \cos\theta_{kn} \qquad k = 1, 2, \cdots, n$$

and the fundamental polynomials of Lagrange interpolation with respect to (1.2) be given by ($x = \cos\theta$):

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$$(1.3)\qquad l_{kn}(\theta)=\frac{(-1)^{k+1}\cos n\theta\sin\theta_{kn}}{n\,(\cos\theta-\cos\theta_{kn})}\qquad k=1,2,\cdots,n.$$

In 1941 G. Grünwald [2] defined a sequence of algebraic polynomials of degree $\leqq n-1$ in x by

$$(1.4)\qquad G_n[f,\theta]=\sum_{k=1}^{n}f(x_{kn})\,A_{kn}(\theta)$$

where

$$(1.5)\qquad 2A_{kn}(\theta)=l_{kn}\left(\theta+\frac{\pi}{2n}\right)+l_{kn}\left(\theta-\frac{\pi}{2n}\right).$$

Moreover he proved that if $f\in C\,[-1,+1]$ then $\lim_{n\to\infty}G_n[f,\theta]=f(\cos\theta)$ and the convergence is uniform in $[-1,+1]$.

The object of this paper is to prove the following theorem.

THEOREM 2. *Let* $f\in C[-1,+1]$ *and we denote by* $\omega(\delta)$ *its modulus of continuity. Then* $(x=\cos\theta)$ *we have*

$$(1.6)\qquad \left|G_n[f,\theta]-f(\cos\theta)\right|=O\left[\omega\left(\frac{\sqrt{1-x^2}}{n}\right)+\omega\left(\frac{1}{n^2}\right)\right].$$

2. Preliminaries

Throughout this section we assume that $(j-1)\pi/n<\theta<j\pi/n$. Moreover, j, k, and i are related by the relation $j+1<k=j+i\leqq n$ or $1\leqq k=j-i<j-1$.

The following inequalities are easy to verify.

$$(2.1)\qquad \sin\frac{\theta}{2}\leqq\sin\frac{\theta}{2}+\sin\frac{(k-1)\pi}{2n}\leqq\sin\frac{\theta}{2}+\sin\frac{k\pi}{2n}$$

$$(2.2)\qquad \sin\frac{\theta_k}{2}\leqq\sin\frac{\theta}{2}+\sin\frac{k\pi}{2n},\qquad \cos\frac{\theta_k}{2}\leqq 2\cos\left(\frac{\theta}{4}+\frac{(k-1)\pi}{4n}\right)$$

$$(2.3)\qquad \cos\frac{\theta}{2}\leqq 2\cos\left(\frac{\theta}{4}+\frac{k\pi}{4n}\right)$$

$$(2.4)\qquad \left|\sin\left(\frac{k\pi}{4n}-\frac{\theta}{4}\right)\right|\geqq\frac{i-1}{2n},\qquad \left|\sin\left(\frac{(k-1)\pi}{4n}-\frac{\theta}{4}\right)\right|\geqq\frac{i-1}{2n}$$

$$(2.5)\qquad \cos\left(\frac{(k-1)\pi}{4n}+\frac{\theta}{4}\right)\geqq\cos\left(\frac{k\pi}{4n}+\frac{\theta}{4}\right)\geqq\frac{i}{2n}\ \text{if}\ \frac{\pi}{2}\leqq\theta\leqq\pi\ \text{and}$$

$$k=\left[\frac{n}{2}\right]+1,\cdots,n$$

$$\cos\left(\frac{(k-1)\pi}{4n}+\frac{\theta}{4}\right) \geqq \cos\left(\frac{k\pi}{4n}+\frac{\theta}{4}\right) \geqq \frac{1}{5} \text{ if } 0 \leqq \theta \leqq \frac{\pi}{2} \text{ and } k = 1,2,\cdots,n \tag{2.6}$$

$$\cos\left(\frac{k-1)\pi}{4n}+\frac{\theta}{4}\right) \geqq \cos\left(\frac{k\pi}{4n}+\frac{\theta}{4}\right) \geqq \frac{1}{5} \text{ if } \frac{\pi}{2} \leqq \theta \leqq \pi, \text{ and } k = 1,2,\cdots,[n/2] \tag{2.7}$$

$$\sin\frac{\theta}{2}+\sin\frac{k\pi}{2n} \geqq \sin\frac{\theta}{2}+\sin\frac{(k-1)\pi}{2n} \geqq \frac{i-1}{n}, \quad 0 \leqq \theta \leqq \frac{\pi}{2} \tag{2.8}$$

$$\sin\frac{\theta}{2}+\sin\frac{k\pi}{2n} \geqq \sin\frac{\theta}{2}+\sin\frac{(k-1)\pi}{2n} \geqq \frac{1}{\sqrt{2}}, \quad \frac{\pi}{2} \leqq \theta \leqq \pi . \tag{2.9}$$

From (1.5) we have another representation of

$$A_k(\theta) = \frac{(-1)^k \sin\theta_k \sin n\theta \sin\theta \sin\frac{\pi}{2n}}{4n\left(\sin^2\frac{k\pi}{2n}-\sin^2\frac{\theta}{2}\right)\left(\sin^2\frac{(k-1)\pi}{2n}-\sin^2\frac{\theta}{2}\right)}. \tag{2.10}$$

Moreover we need

$$A_k(\theta)+A_{k+1}(\theta) = \frac{\left(\sin^2\frac{(k+1)\pi}{2n}-\sin^2\frac{(k-1)\pi}{2n}\right)A_k(\theta)}{\sin^2\frac{(k+1)\pi}{2n}-\sin^2\frac{\theta}{2}} + \frac{\sin\frac{\pi}{2n}A_{k+1}(\theta)}{\sin\theta_{k+1}}, \tag{2.11}$$

$$\left|\frac{\sin^2\frac{(k+1)\pi}{2n}-\sin^2\frac{(k-1)\pi}{2n}}{\sin^2\frac{(k+1)\pi}{2n}-\sin^2\frac{\theta}{2}}\right| \leqq \frac{2\pi}{i}. \tag{2.12}$$

3. Estimates

The following estimates of $A_k(\theta)$ are needed for the proof of Theorem 2.

Lemma 1. *If $(j-1)\pi/n < \theta < j\pi/n$, letting $j+1 \leqq k = j+i \leqq n$ or $1 \leqq k = j-i < j-1$, then we have:*

$$|A_k(\theta)| = O(1) \text{ if } k = j-1, j, \text{ or } j+1, \tag{3.1}$$

$$|A_k(\theta)| \leqq \frac{\pi}{i^2} \tag{3.2}$$

$$|A_k(\theta)| \leqq \frac{5\pi n \sin\theta}{i^3}, \tag{3.3}$$

$$(3.4)\qquad |A_k(\theta)| \leqq \frac{5\pi n \sin\theta_k}{i^3},$$

$$(3.5)\qquad |A_k(\theta)| \leqq \frac{25n^2 \sin\theta \sin\theta_k}{i^4}.$$

PROOF. For (3.1) see [2]. (3.2) follows from (2.10), (2.1)-(2.4). If $0 \leqq \theta \leqq \pi/2$ then (3.3) follows from (2.10), (2.2), (2.4), (2.6), and (2.8). If $\pi/2 \leqq \theta < \pi$, then (3.3) follows from (2.10), (2.4), (2.5), (2.7), and (2.9). Proofs for (3.4) and (3.5) are based on the same lines.

LEMMA 2. *Let $(j-1)\pi/n < \theta < j\pi/n$. Then we have:*

$$(3.6)\qquad |A_k(\theta) + A_{k+1}(\theta)| = \frac{5\pi^2}{i^3} \text{ if } j+1 < k = j+i < n,$$

$$(3.7)\qquad |A_k(\theta) + A_{k-1}(\theta)| = \frac{5\pi^2}{i^3} \text{ if } 1 < k = j-i < j-1.$$

Also we have

$$(3.8)\qquad |A_k(\theta) + A_{k+1}(\theta)| = \frac{360\, n \sin\theta}{i^4} \text{ if } j+1 < k = j+i < n,$$

$$(3.9)\qquad |A_k(\theta) + A_{k-1}(\theta)| = \frac{360\, n \sin\theta}{i^4} \text{ if } 1 < k = j-i < j-1.$$

PROOF. (3.6) follows immediately from (2.11), (2.13), (3.2), and (3.4). (3.8) is a simple consequence of (2.11), (3.3), and (3.5). Proof of (3.7) and (3.9) are similar so we omit the details.

It is easy to verify that $j/n \leqq \sin\theta \leqq j\pi/n$ for $j = 2, 3, \cdots, [n/2]$, $(n-j)/n \leqq \sin\theta \leqq (n-j)\pi/n$ for $j = [n/2]+1, \cdots, n-1$. On using these results and known properties of modulus of continuity we have

$$\omega\left(\frac{i}{n^2}\right) \leqq \frac{2i}{j}\,\omega\left(\frac{\sin\theta}{n}\right) \qquad j \leqq \left[\frac{n}{2}\right]$$

$$\omega\left(\frac{i}{n^2}\right) \leqq \frac{2i}{n-j}\,\omega\left(\frac{\sin\theta}{n}\right) \qquad j \geqq \left[\frac{n}{2}\right].$$

With the help of these results we recast the two lemmas of O. Kis [3] as they are needed in the proof.

LEMMA 3 (O. Kis [3]). *Let $(j-1)\pi/n < \theta < j\pi/n$. Then we have*

$$|f(x_k) - f(x)| = O\left[\omega\left(\frac{\sin\theta}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right] \qquad \text{if } k = j$$

$$= O\left[\frac{i^2}{j}\omega\left(\frac{\sin\theta}{n}\right)\right] \quad \text{if } j \leqq \left[\frac{n}{2}\right]$$

$$= O\left[\frac{i^2}{n-j}\omega\left(\frac{\sin\theta}{n}\right)\right] \quad \text{if } j > \left[\frac{n}{2}\right].$$

LEMMA 4 (O. Kis [3]). *Let* $(j-1)\pi/n < \theta < j\pi/n$. *Then*

$$|f(x_k) - f(x_{k+1})| = O\left[\frac{i}{j}\omega\left(\frac{\sin\theta}{n}\right)\right] \quad \textit{if } j \leqq \left[\frac{n}{2}\right]$$

$$= O\left[\frac{i}{n-j}\omega\left(\frac{\sin\theta}{n}\right)\right] \quad \textit{if } j > \left[\frac{n}{2}\right].$$

Similar estimates hold for $|f(x_k) - f(x_{k-1})|$.

PROOF OF THEOREM 2. By (1.5) and the definition of θ we have

$$G_n[f,\theta] - f(\cos\theta) = \sum_{k=1}^{n} (f(x_k) - f(x))\, A_k(\theta)$$

$$= \sum_{k=1}^{j-2} + u_{j-1} + u_j + u_{j+1} + \sum_{k=j+2}^{n}.$$

Of course, if $j = 1$ or 2 (or $n-1, n$) the first (or last) summation will not appear. Estimates of $\sum_{k=1}^{j-2}$ and $\sum_{k=j+2}^{n}$ are similar.

By (3.10) and (3.1) we have

$$|u_j| = |f(x_j) - f(x)|\ |A_j(\theta)| = O\left[\omega\left(\frac{\sin\theta}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right].$$

Similarly the estimates of u_{j-1} and u_{j+1} can be computed. Let us now estimate

$$T \equiv \sum_{k=j+2} (f x_k) - f(x)) A_k(\theta).$$

Here we use extremely useful ideas of O. Kis [3], and group the summands in pairs. If we set

$$B_k(\theta) \equiv (f(x_k) - f(x))\, A_k(\theta) + (f(x_{k+1}) - f(x))\, A_{k+1}(\theta)$$

$$= (A_k(\theta) + A_{k+1}(\theta))\, (f(x_k) - f(x)) + (f(x_{k+1}) - f(x))\, A_{k+1}(\theta)$$

then we obtain $T = \sum_{i \in I} B_{j+i}(\theta) + [A_n(\theta)\ \ (f(x_n) - f(x))]$ where

$$I = \{i:\ i < n - j, \quad i = 2, 4, 6, \cdots\}.$$

The last term written in the bracket is to signify that it appears only if $n - j$ is even. Let $0 \leqq \theta \leqq \pi/2$; then we express

$$\Sigma|B_{j+i}(\theta)| = \sum_{i\leqq j} |B_{j+i}(\theta) + \sum_{i>j} |B_{j+i}(\theta)|,$$

and use appropriate parts of Lemmas 1,2,3, and 4. Similarly, in the case $\pi/2 \leqq \theta \leqq \pi$, we express

$$\Sigma|B_{j+i}(\theta)| = \sum_{i\leqq n-j} |B_{j+i}(\theta)| + \sum_{i\geqq n-j} |B_{j+i}(\theta)|$$

and use once again Lemmas 1,2,3, and 4 which gives the desired result. This proves the theorem.

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